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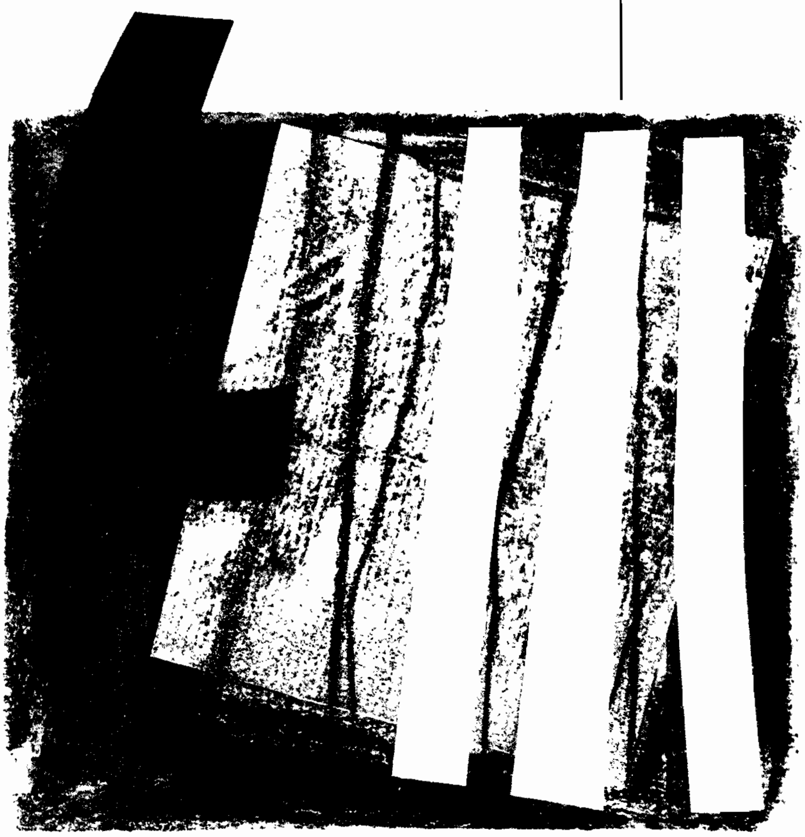

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Abstract

In this note a conjecture of Nagaraja and Nevzorov (1997) concerning the characterization of distributions functions by a convex conditional mean function is proved. It is not neccesary furthermore to suppose that the c.d.f. is continuous and the characterization does hold without assumptions considered in the result of the above reference. The present note gives a general theorem of characterization in terms of the *convex conditional mean function* introduced by Nagaraja and Nevzorov (1997) determining explicitly the distribution function associated with each one.

Keywords: Mean lifetime function; mean deathtime function; convex conditional mean; conditioned lifetime function; characterization of distributions.

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NOTE ON A CHARACTERIZATION PROBLEM OF

NAGARAJA AND NEVZOROV

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ABSTRACT

In this note a conjecture of Nagaraja and Nevzorov (1997) concerning the characterization of distributions functions by a convex conditional mean function is proved. It is not neccesary furthermore to suppose that the c.d.f. is continuous and the characterization does hold without assumptions considered in the result of the above reference. The present note gives a general theorem of characterization in terms of the *convex conditional mean function* introduced by Nagaraja and Nevzorov (1997) determining explicitly the distribution function associated with each one.

Key Words: Mean lifetime function; Mean deathtime function; Convex conditional Mean conditioned lifetime function; characterization of distributions.

AMS Clasification: 62E10.

1. INTRODUCTION

The problem considered in this, paper is inspired on the conjecture of Nagaraja and Nevzorov (1997) pointed out by one of the referees in the Remark 3 of their paper. The key question is whether the *convex conditional mean function* (CCMF)

$$M(x) = \alpha E[X \mid X \leq x] + (1-\alpha)E[X \mid X > x] \tag{1}$$

uniquely identifies the c.d.f. F related to the random variable X. The answer is 'yes', without the need of assuming the continuity of F. Specifically the result proved by Nagaraja and Nevzorov (1997) is the following:

Theorem 1 *Let X and Y be two random variables with support (a, b), $-\infty \leq a < b \leq \infty$, and continuous c.d.f.s F and G, respectively. Assume that $E[X]$ exists, and there exists a* *point x_0 such that for some α, $0 < \alpha < 1$, the following conditions are satisfied:*

$$F(x_0) = G(x_0) = \alpha \tag{2}$$

and

$$E[X \mid X \leq x_0] = E[Y \mid Y \leq x_0] \tag{3}$$

Then $F = G$ if and only if

$$\alpha E[X \mid X \leq x] + (1-\alpha) E[X \mid X > x] = \alpha E[Y \mid Y \leq x] + (1-\alpha) E[Y \mid Y > x] \tag{4}$$

is satisfied for all x in (a, b).

Indeed, Nagaraja and Nevzorov (1997) considered distributions functions with closed interval as support instead of the open interval stated in Theorem 1. However the calculations and development in this paper are more direct using the open interval although they do hold in the other case. From now on, $g(a)$ and $g(b)$ will denote $\lim_{y \to a^+} g(y)$ and $\lim_{y \to b^-} g(x)$, respectively for any other function $g(x)$ used here.

Nagaraja and Nevzorov (1997) remarked that when $M(x)$ has a simple form, F can be determined explicitly and in such cases, (2) and (3) will be clearly unnecessary. They also conjectured that the characterization would do hold even without these assumptions. We will prove this conjecture and therefore, assumptions (2) and (3) are always unnecessary. The characterization of F by the *convex conditional mean function* will be based on two characterization results of distributions in terms of conditional expectations. Here we provide a brief description of the context that we will treat, which is intended to make the paper relatively self–contained. We do not discuss many of the concepts in detail, since the background of this note is given in Balkema (1974), Galambos and Kotz (1978), Hamdan (1972), Kotlarski and Hinds (1975) and Lillo and Martín (1997).

Consider a random variable X with support the interval (a, b), $-\infty \leq a < b \leq \infty$ and finite mean. Then

$$m(x) = E[X \mid X > x], \text{ for } a < x < b \tag{5}$$

is called the *mean lifetime function*, (MLF), and

$$d(x) = E[X \mid X \leq x], \text{ for } a < x < b \tag{6}$$

is called the *mean deathtime function*, (MDF).

With this notation and concepts, the paper is organized as follows. Section 2 contains two results of characterization of the c.d.f F in terms of the MLF and the MDF, respectiveley. In Section 3 the key question of this note is answered since the CCMF uniquely identifies the c.d.f F. In the final part of the paper, we apply all of these results to the uniform distribution.

2. TWO RESULTS OF CHARACTERIZATIONS

A natural question is: what are the necessary and sufficient conditions to guarantee that a function from (a, b) onto the real line will be a MLF or a MDF?. This paper gives a fairly complete answer to this question deriving furthermore the c.d.f. F related to these conditional expectations. For any distribution function F, denote $\bar{F} = 1 - F$. In the following, we will use a lemma based on a result of Kotz and Shanbhag (1980).

Lemma 1 *Let F be any distribution function on (a, b), with finite first moment. Then*

$$\bar{F}(x) = \lim_{y \to a} \frac{r(y)}{r(x)} \exp\left\{-\int_y^x \frac{dt}{r(t)}\right\} \tag{7}$$

where $r(x) = m(x) - x$ and m was defined in (5).

Meilijson (1972) proved Lemma 1 for c.d.f.s with support in $(0, \infty)$. From now on, we will denote by $\mathcal{F}_{(a,b)}$ the class of c.d.f.s with support in (a, b) and finite mean. Firstly, we consider the problem of identification of the MLF related to functions belonging class $\mathcal{F}_{(a,b)}$. Shanbhag (1970) characterized the exponential distribution in terms of the mean lifetime function and, Hamdan (1972) gives a characterization for a kind of absolutely continuous distributions involving conditional expectations. The most general way of this kind of characterizations of continuous distributions, is given by Kotlarski and Hinds (1975). Kotz and Shanbhag (1980) developed some approaches to the characterization of real-valued random variables without restricting to positive random variables and to those having infinite right extremity, extending the results of Hamdan (1972) and Gupta (1975). In the next result, we characterize any distribution through its MLF deriving furthermore the c.d.f. associated with it.

Theorem 2 *Let m be a function from (a, b), $-\infty \leq a < b \leq \infty$ with $-\infty < m(a)$. Then, m is a MLF for some $F \in \mathcal{F}_{(a,b)}$ if and only if*

1. *$m(x)$ is a right-continuous fuction and $m(x) > x$ for all $a < x < b$.*
2. *$m(x)$ is a non-decreasing funçtion for all $a < x < b$.*
3. *For all $a < x < b$*

$$\lim_{y \to a} \int_y^x \frac{dt}{m(t) - t} \begin{cases} < \infty & \text{if} \quad a > -\infty \\ = \infty & \text{if} \quad a = -\infty \end{cases}$$

4. *Taking limits as $x \to b$*

$$\int_a^b \frac{dt}{m(t) - t} = \infty$$

Moreover, let m be a function fulfilling these four properties, then there exists a unique function $F \in \mathcal{F}_{(a,b)}$ such that $m_F = m$, where m_F denotes the MLF related to the c.d.f. F.

In this case, F is determined as

$$\bar{F}(x) = \frac{m(a)-a}{m(x)-x}\exp\left\{-\int_a^x \frac{dt}{m(t)-t}\right\},\ \text{if}\quad a>-\infty \tag{8}$$

$$\bar{F}(x) = \lim_{y\to-\infty}\frac{m(y)-y}{m(x)-x}\exp\left\{-\int_y^x \frac{dt}{m(t)-t}\right\},\ \text{if}\quad a=-\infty \tag{9}$$

Proof: Firstly, we assume that m is the MLF related to some distribution function F within class $\mathcal{F}_{(a,b)}$ with $-\infty < a$. From the definition of MLF given in (5) it is immediate that $m(x) > x$ and it may be deduced easily that a $MRLF$ is right–continuous. Since $m(x)$ can be stated in connection with $\bar{F}(x)$, we have for $a < x < b$

$$\begin{aligned} m(x+h)-m(x) &= \frac{\int_{x+h}^{\infty}\bar{F}(t)dt}{\bar{F}(x+h)} - \frac{\int_x^{\infty}\bar{F}(t)dt}{\bar{F}(x)} + h \\ &\geq \frac{h\bar{F}(x)-\int_x^{x+h}\bar{F}(t)dt}{\bar{F}(x)},\ \text{for all } h>0 \end{aligned} \tag{10}$$

Since $\bar{F}(x)$ is non–increasing (10) is greater than zero and then, $m(x)$ is non-decreasing. To verify the third condition, observe that

$$\int_a^x \frac{dt}{m(t)-t} = \int_a^x \frac{\bar{F}(t)}{\int_t^b \bar{F}(y)dy}dt$$

Putting

$$h(t) = \int_t^b \bar{F}(y)dy \tag{11}$$

we can write

$$\int_a^x \frac{dt}{m(t)-t} = \ln\left[h(a)\right] - \ln\left[h(x)\right]$$

Since $F \in \mathcal{F}_{(a,b)}$, the above difference is finite. Condition 4 yields considering that

$$\lim_{t\to b} h(t) = 0$$

Hence, from Lemma 1, $\bar{F}(x)$ can be rewritten as in (8). Now, we prove the sufficient conditions. To this end, we will show that function $\bar{F}$ defined in (8) determines a distribution function in (a,b) with finite mean. To avoid heavy notation, we will use $r(x) = m(x) - x$. Obviously $\bar{F}(a) = 1$. Now, we have to prove that $\lim_{x\to b}\bar{F}(x) = 0$ which is true iff

$$\lim_{x\to b} r(x)e^{\int_a^x \frac{dt}{r(t)}} = \infty$$

By Condition 2, it is easy to see that

$$\lim_{x\to b} r(x)e^{\int_a^x \frac{dt}{r(t)}} > \begin{cases} \lim_{x\to b} m(x)e^{\int_a^x \frac{dt}{r(t)}} & \text{if}\quad b\geq 0 \\ \lim_{x\to b}\left[m(a)-x\right]e^{\int_a^x \frac{dt}{r(t)}} & \text{if}\quad b<0 \end{cases} \tag{12}$$

Using condition 4, both limits in (12) are infinite and hence, $\bar{F}(b) = 0$. On the other hand, we may prove that $\bar{F}(t)$ is a non–increasing function of t which it is true iff

$$\frac{\bar{F}(x)}{\bar{F}(x+h)} = \frac{r(x+h)}{r(x)} e^{\int_x^{x+h} \frac{dt}{r(t)}} \geq 1 \quad \text{for } h \geq 0 \quad a < x < b.$$

To prove this, note that

$$\begin{aligned} \frac{\bar{F}(x)}{\bar{F}(x+h)} &\geq \frac{r(x+h)}{r(x)} e^{\int_x^{x+h} \frac{dt}{m(x+h)-t}} \\ &= \frac{m(x+h)-x}{r(x)} \end{aligned} \tag{13}$$

Now, it is easy to deduce that (13) is greater than one, using the second condition. Finally, the expected value associated with the distribution derived from $\bar{F}$ is $r(a)$, which is finite. Thus, we have proved that $F \in \mathcal{F}_{(a,b)}$. Now we only have to show that $m_F = m$ or equivalently $r_F(x) = r(x)$. By definition,

$$r_F(x) = \frac{\int_x^b \frac{r(a)}{r(t)} \exp\left\{-\int_a^t \frac{dy}{r(y)}\right\} dt}{\bar{F}(x)}$$

Hence, taking

$$z(x) = \int_a^x \frac{dy}{r(y)} \Longrightarrow \frac{dz(x)}{dx} = \frac{1}{r(x)}$$

we arrive at

$$r_F(x) = \frac{r(a)e^{-z(x)}}{\frac{r(a)}{r(x)}e^{-z(x)}} = r(x)$$

Thus the proof is complete since $r_F = r$ implies that $m_F = m$. Now, assume that $a = -\infty$, then the necessary conditions are immediate from (7). To establish the second part, consider

$$\bar{F}_y(x) = \frac{m(y)-y}{m(x)-x} \exp\left\{-\int_a^x \frac{dt}{m(t)-t}\right\}, \quad \text{for all} \quad -\infty < y$$

and note that F_y is a distribution function on (y, b). Let $m_y(x)$ denote its MRL associated. It is easy to see that as $y \to b$, $m_y(x) \to m(x)$ for every continuity point of m and $\bar{F}_y(x) \downarrow \bar{F}(x)$, with F as stated in (9). As $\lim_{y \to a} m(y) > \infty$, $\bar{F}$ is a Borel–measurable function integrable with respect to the Lebesgue measure and then, from a stability theorem concerning MRL functions given by Kotz and Shanbhag ((1980), Proposition 4, pp. 911), $m_F = m$ and the proof is complete. ∎

Following a development similar to prove Theorem 2, we can also characterize distributions in terms of its MDF such as it is shown in the next theorem.

Theorem 3 *Let d be a function from (a,b), $-\infty \leq a < b \leq \infty$ with $\lim_{y\to b} m(y) < \infty$. Then, d is a MDF for some $F \in \mathcal{F}_{(a,b)}$ if and only if*

1. *$d(x)$ is a right-continuous fuction and $d(x) < x$ for all $a < x < b$.*

2. *$d(x)$ is a non-decreasing function for all $a < x < b$.*

3. *For all $a < x < b$*

$$\lim_{y\to b} \int_x^y \frac{dt}{t - d(t)} \begin{cases} < \infty & \text{if} \quad b < \infty \\ = \infty & \text{if} \quad b = \infty \end{cases}$$

4. *Taking limits as $x \to a$*

$$\int_a^b \frac{dt}{t - d(t)} = \infty$$

Moreover, let d be a function fulfilling these four properties, then there exists a unique function $F \in \mathcal{F}_{(a,b)}$ such that $d_F = d$, where d_F denotes the MDF related to the c.d.f. F. In this case, F is determined as

$$F(x) = \frac{b - d(b)}{x - d(x)} \exp\left\{ - \int_x^b \frac{dt}{t - d(t)} \right\} \text{ if } \quad b < \infty \tag{14}$$

$$F(x) = \lim_{y\to b} \frac{y - d(y)}{x - d(x)} \exp\left\{ - \int_x^y \frac{dt}{t - d(t)} \right\} \text{ if } \quad b = \infty \tag{15}$$

We omit the proof since it is based on the same arguments as the proof of Theorem 2. We want to point out that we have proved Theorem 2 in reference Lillo and Martín (1997) within a Bayesian framework, in terms of the *mean residual lifetime function* and applied only to positive random variables with infinite essential supremum.

3. CHARACTERIZATION BY THE CONVEX CONDITIONAL MEAN

Observe that (1) can be rewritten in terms of the MLF and the MDF as

$$M(x) = \alpha d(x) + (1 - \alpha) m(x) \tag{16}$$

Now the question is whether $M(x)$ characterizes completely to the c.d.f F. A generalization of the partial characterization given by Nagaraja and Nevzorov (1997) (see Theorem 1) is the next theorem in which the distribution function can be determined explicitly

Theorem 4 *Fixed α, $0 < \alpha < 1$. A function $M(x)$ from (a,b) is a convex conditional mean for some $F \in \mathcal{F}_{(a,b)}$ if and only if there exists two functions $m(x)$ and $d(x)$ from (a,b) such that*

$$M(x) = \alpha d(x) + (1 - \alpha) m(x) \tag{17}$$

where $d(x)$ is a MDF, $m(x)$ is a MLF and for $a < x < b$

$$\lim_{y\to b}\frac{y-d(y)}{x-d(x)}\exp\left\{-\int_x^y\frac{dt}{t-d(t)}\right\} = 1-\lim_{z\to a}\frac{m(z)-z}{m(x)-x}\exp\left\{-\int_z^x\frac{dt}{m(t)-t}\right\} \tag{18}$$

In this case, the c.d.f F is uniquely determined by (18) such that

$$F(x) = \lim_{y\to b}\frac{y-d(y)}{x-d(x)}\exp\left\{-\int_x^y\frac{dt}{t-d(t)}\right\} \tag{19}$$

Proof: After observing (16), it is obvious that the decomposition is necessary. From Theorem 2 and Theorem 3 we can easily identify whether d and m are MDF and MLF respectively. Condition 18 ensures that the c.d.f.s corresponding with d and m drawn in (14) and (8) are the same. Finally, we only have to prove the uniqueness of the decomposition stated in (17) if, it is possible. Assume that there exists two pairs (d, m) and (d', m') such that

$$M(x) = \alpha d(x) + (1-\alpha)m(x) = \alpha d'(x) + (1-\alpha)m'(x) \quad \text{for all} \quad a \le x \le b \tag{20}$$

and besides, assume that (d, d') are MDF and (m, m') are MLF. Since, we proved that both MLF and MDF characterize the distribution, there exists two c.d.f.s F and F' related to (d, m) and (d', m') respectively but $F \neq F'$. Let x_0 be,

$$x_0 = \inf\{\, x \,/\, F(x) \neq F'(x)\,\} \tag{21}$$

First, assume that $x_0 > a$. Since $\mu = \lim_{x\to a} M(x)$ is the expected value related to both F and F' and due to

$$m(x) = \frac{\mu - \int_a^x t\,dF(t)}{\bar{F}(x)}$$

we have that $m(x) = m'(x)$ for all $a \le x < x_0$ and suppose, for example, that $m(x_0) > m'(x_0)$. By definition of d, it is obvious that $d(x) = d'(x)$ for all $a \le x < x_0$. Note that if we show that $d(x_0) > d'(x_0)$ then (20) is not true for all $t \in (a, b)$ and the proof would be complete. Then, $m(x_0) > m'(x_0)$ implies

$$\frac{1}{m(x_0)-x_0}e^{-\int_a^{x_0}\frac{dt}{m(t)-t}} < \frac{1}{m'(x_0)-x_0}e^{-\int_a^{x_0}\frac{dt}{m'(t)-t}}$$

Since (18) does hold for both pairs (d, m) and (d', m'), we have

$$\frac{1}{x_0-d(x_0)}e^{-\int_{x_0}^b\frac{dt}{t-d(t)}} < \frac{1}{x_0-d'(x_0)}e^{-\int_{x_0}^b\frac{dt}{t-d'(t)}}$$

and as $d(x) = d'(x)$, $a \le x < x_0$, the integral is the same in two parts of the above inequality and then, $d(x_0) > d'(x_0)$. Now, if $x_0 = a$, the proof is different. Assume that $F(x) < F'(x)$ for all $x \in (a, y_0)$. It is obvious that y_0 exists and that $m(x) < m'(x)$, $x \in (a, y_0)$. Let x belong previous interval, then

$$m'(x) - m(x) = \gamma > 0 \quad F'(x) - F(x) = \beta > 0$$

Now, the objective is to prove that $M(x) < M'(x)$. By straighforward manipulations we have,

$$M'(x) - M(x) > 0 \Longleftrightarrow -\alpha\beta\mu - \alpha F(x)m'(x) + \alpha F'(x)m(x) + F(x)F'(x)\gamma > 0 \qquad (22)$$

which since can be rewritten as

$$M'(x) - M(x) = \alpha\left[F'(x)\left(m(x) - \mu\right) + F(x)\mu\right] + F(x)F'(x)\gamma$$

Hence, if $\mu \geq 0$ (22) is fulfilled considering that $m(x) \geq \mu$ since m is a non–decreasing function. If $\mu < 0$, we can choose x such that $m'(x) < 0$, then

$$M'(x) - M(x) = \alpha\beta\left(m(x) - \mu\right) - \alpha F(x)m'(x) + F(x)F'(x)\gamma$$

and (22) is also true and the proof is complete. ∎

Remark 1 Theorem 4 implies that a convex conditional mean function is uniquely identified by a pair (α, d) where d is a MDF and $0 < \alpha < 1$. To show this fact, it is sufficient consider that the c.d.f. F is given by (19) and the relation

$$M(x) = \left[\frac{\alpha - (1-\alpha)F(x)}{\bar{F}(x)}\right] d(x) + (1-\alpha)\frac{\mu}{\bar{F}(x)}$$

Equivalently a CCMF can be uniquely involved by a pair (α, m) where m is a MLF.

4. UNIFORM DISTRIBUTION

As an application of the results involved in previous sections, we give a characterization of the uniform distribution in terms of the *convex conditional mean function*.

Theorem 5 *$M(x) = \lambda x + k$ is a CCML in (a, b) if and only if $\lambda = 1/2$ and $a/2 < k < b/2$. In this case, the c.d.f F is uniform over the interval (a, b).*

Proof: For any α, $0 < \alpha < 1$, we have the following decomposition

$$M(x) = \alpha\left[\lambda x + \frac{k - (1-\alpha)(1-\lambda)b}{\alpha}\right] + (1-\alpha)\left[\lambda x + (1-\lambda)b\right] \qquad (23)$$

By Theorem 2, we obtain that $m(x) = \lambda x + (1-\lambda)b$ is a MLF with associated c.d.f

$$\bar{F}_m(x) = \left(\frac{b-t}{b-a}\right)^{\frac{\lambda}{1-\lambda}} \qquad (24)$$

By Theorem 3, $d(x) = \lambda x + (k - (1-\alpha)(1-\lambda)b)(\alpha)^{-1}$ is a MDF with c.d.f

$$\bar{F}_d(x) = 1 - \left(\frac{x - \frac{1}{\alpha}\left(\frac{k}{1-\lambda} - (1-\alpha)b\right)}{b - \frac{1}{\alpha}\left(\frac{k}{1-\lambda} - (1-\alpha)b\right)}\right)^{\frac{\lambda}{1-\lambda}} \qquad (25)$$

Since $\bar{F}_d(a) = 1$, the value of α is such that

$$\alpha = \frac{b - 2k}{b - a} \tag{26}$$

Impossing that $0 < \alpha < 1$, we obtain the condition $a/2 < k < b/2$ and considering the condition (18) of Theorem 4 for (24) and (25), the second assumption follows; that is, $\lambda = 1/2$. The uniquiness of Theorem 4 ensures that in this case, the unique possible distribution is uniform in (a, b). ■

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